DEPARTMENT OF PHYSICS AND ASTROPHYSICS
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B.Sc.(H) PHYSICS (CBCS)
SEMESTER - VI

STATISTICAL MECHANICS

Model Problem Set

1 Classical Statistics

1.1 Macrostate and Microstate, Phase Space, Ensemble, Thermodynamic Probability

Problem 1: Consider an isolated system of four non-interacting spins labelled 1, 2, 3, and 4, each with magnetic moment m, interacting with an external magnetic field B. Each spin can be parallel ('up') or antiparallel ('down') to B, with the energy of a spin parallel to B equal to $\epsilon = -mB$ and the energy of a spin antiparallel to B equal to $\epsilon = +mB$. Let the total energy of the system be $E = -2mB$.

(a) How many microstates of the system correspond to this macrostate? Enumerate these microstates.

(b) What is the probability that the system is in a given microstate in equilibrium?

(c) What is the probability that a given spin points up? Use this probability to campute the mean magnetic moment of a given spin in equilibrium.

(d) What is the probability that if spin 1 is 'up', spin 2 is also 'up'?

Problem 2: Consider a system of four non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, ...$ The system is divided into two subsystems A and B, subsystem A consisting of particles 1 and 2, and B consisting of particles 3 and 4 respectively. A and B are initially thermally insulated from each other, with energies $E_A = 5\epsilon_0$ and $E_B = \epsilon_0$. What are the possible microstates of the composite system? Now, suppose the two subsystems are allowed to thermally interact with each other, so that they can exchange energy without the total energy of the system changing. After equilibrium is attained, enumerate the possible microstates of the composite system. In equilibrium, what is the probability that subsystem A has energy E_A, for $E_A = 0, \epsilon_0, 2\epsilon_0, .., 6\epsilon_0$? For what value of E_A is the probability maximum?

1.2 Entropy and Thermodynamic Probability

Problem 1: Consider a system of N particles (which could be interacting with each other) with energy E and occupying a volume V. The entropy of the system is known to be extensive. Suppose the energy of the system is changed, such that the new energy is λE, where λ is a multiplicative factor. Can you say that the new entropy will be λS, where S is the original entropy? If not, what other changes will be needed such that this is true?

Problem 2: Consider a system of $N >> 1$ weakly interacting particles, each of which can be in quantum states with energies $0, \epsilon, 2\epsilon, 3\epsilon, ...$ Given the system has a certain energy, the temperature of the system is given by

$$\begin{aligned} \frac{1}{T} &= \frac{\partial S}{\partial E} \\ &\simeq \frac{\Delta S}{\Delta E} \end{aligned}$$

where ΔS is the change in the entropy of the system due to the change in the energy of the system by ΔE.

(a) If the system is in its ground state, what is its entropy?

(b) If the total energy of the system is ϵ, what is its entropy?

(c) What is the change in entropy of the system if the total energy of the system is increased from ϵ to 2ϵ?

(d) Given the above definition of temperature, what is the temperature of the system if its total energy is ϵ?

Problem 3: A system of four weakly interacting distinct particles is such that each particle can be in one of four states with energies $\epsilon, 2\epsilon, 3\epsilon$ and 4ϵ respectively. If the system has total energy 15ϵ, what is the entropy of the system? For what possible values of total energy is the entropy of the system zero?

Problem 4: Consider a lattice of N non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, ...$ The system is in equilibrium.

(a) If the energy of the system is E, what is the number of microstates of the system?

(b) Find an expression for the entropy of the system as a function of energy and simplify it using Sterling's approximation $\ln n \simeq n \ln n - n$ for $n >> 1$.

(c) Using the relation

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

determine a relation between the energy of the system and its temperature.

Hint: The problem of determining the number of microstates can be reduced to counting the number of ways of arranging a certain number of sticks and a certain number of dots along a line.

1.3 Maxwell Boltzmann Distribution

Problem 1: Consider atomic hydrogen in thermal equilibrium at temperature T. Estimate the ratio of the number of atoms with energy $E = -3.4$ eV to the number of atoms with energy $E = -13.6$ eV for $T = 1000°K$.

Problem 2: A system of N weakly interacting particles, each of mass m, is in thermal equilibrium at temperature T. The system is contained in a cubical box of side L, whose top and bottom surfaces are parallel to the Earth's surface, where the acceleration due to gravity is g. A coordinate system is set up with the origin at the centre of the base of the box and the positive z axis along the vertical direction, such that the ranges of coordinates accessible to any particle are $-L/2 \leq x \leq L/2$, $-L/2 \leq y \leq L/2$, $0 \leq z \leq L$.

(a) What is the probability that a given particle has velocity in the range (v_x, v_y, v_z) and $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$?

(b) What is the probability that a given particle has x coordinate between x and $x + dx$?

(c) What is the probability that a given particle has y coordinate between y and $y + dy$?

(d) What is the probability that a given particle has z coordinate between z and $z + dz$?

(e) From the above probability distributions, calculate the mean kinetic and potential energies of a particle.

Problem 3: A sensitive spring balance consists of a quartz spring with spring constant k. This balance is used to measure the mass of very tiny, light objects by suspending them from the balance and observing the extension in the spring. Consider a tiny object of mass m suspended from the spring. The object is in an environment which is at temperature T, and gets 'kicked' around by it, reaching equilibrium with the environment.

1. What is the potential energy of the system if the spring is extended by x?
2. What is the probability that the spring is extended by x relative to its equilibrium length?
3. Calculate the mean extension $\overline{x}$ and the mean squared extension $\overline{(x-\overline{x})^2}$.
4. Comparing the square root of the mean squared extension with the mean extension, extimate the minimum mass that can be reliably measured.

Problem 4: A one-dimensional chain consists of a set of N rods each of length a. Each rod can align either parallel or perpendicular to the length of the chain independently of its neighbours. The energy of the rod is $-\epsilon$ when it is perpendicular to the length of the chain and is ϵ when the rod is parallel to the length of the chain. Show that at temperature T, the approximate average length of the chain is $Na/(1+e^{2\beta\epsilon})$, where $\beta = 1/k_B T$. What is the chain length at very high T?

1.4 Partition Function, Heat Capacity, Entropy, Saha's Ionisation Potential

Problem 1: Consider a single particle system with five states. There is one state with energy 0, two states with energy ϵ and two states with energy 2ϵ. The system is in equilibrium with a heat bath at temperature T.

(a) Calculate the partition function for the system.

(b) Calculate the mean energy and heat capacity of the system as functions of temperature.

(c) What is the relative probability of the system having energy 2ϵ and ϵ?

Problem 2: The partition function of a system is given by

$$lnZ = aT^4V$$

where T is the absolute temperature, V is the volume of the system and a is a constant. Evaluate the mean energy, pressure and entropy of the system.

Problem 3: Consider a simplified model of graphite, in which each carbon atom acts as a harmonic oscillator, oscillating with frequency ω within the layer and frequency ω' perpendicular to it. The oscillations in the three directions are independent, such that the expression for energy of a carbon atom is

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{m}{2}(\omega^2 x^2 + \omega^2 y^2 + \omega'^2 z^2)$$

where coordinates x, y are in the plane of the layer and z is perpendicular to it. The sample is at temperature T, such that $\hbar\omega >> T$ and $\hbar\omega' << T$ (the restoring forces in the plane of the layer are much stronger than those perpendicular to it).

(a) Given the temperature conditions, one kind of the oscillations (in the plane or perpendicular to it) can be treated classically, and the other quantum mechanically with only the ground and first excited states appreciably populated. Identify the corresponding oscillations.

(b) Taking into account the above considerations, calculate the partition function and show that it factorises into three factors, two of which are identical.

(c) Find an expression for the molar specific heat of the system as a function of temperature, using approximations appropriate to the temperature conditions stated above.

Problem 4: N diatomic molecules are stuck on a surface. Each molecule can either lie flat on the surface (in which case it can orient itself either along the x or the y direction) or it can stand up perpendicular to the surface (along the z direction). Assume that the flat configurations have zero energy and the configuration perpendicular to the surface has energy $\epsilon > 0$. The system is in thermal equilibrium at temperature $T > 0$.

(a) Calculate the partition function of the system.

(b) Calculate the mean energy of the system. What is the largest possible value for this energy (attained by changing the temperature)?

(c) Calculate the heat capacity of the system as a function of temperature.

(d) What is the probability of a given molecule 'standing up'?

Problem 5: Four moles of nitrogen and one mole of oxygen at $P = 1\text{atm}$ and $T = 300^0\text{K}$ are mixed together to form a mixture at the same pressure and temperature. Calculate the entropy of mixing per mole of the mixture formed.

Problem 6: The center of a star has matter at density $10^5 kgm^{-3}$, at temperature $1.5 \times 10^7\text{K}$. What fraction of H is ionised under these conditions?

1.5 Negative Temperatures

Problem 1: Consider an isolated system of $N >> 1$ weakly interacting, distinct particles in equilibrium. Each particle can be in one of three states, with energies $0, \epsilon$ and 2ϵ respectively. Given the system has a certain energy, the temperature of the system is given by

$$\begin{aligned} \frac{1}{T} &= \frac{\partial S}{\partial E} \\ &\simeq \frac{\Delta S}{\Delta E} \end{aligned}$$

where ΔS is the change in the entropy of the system due to the change in the energy of the system by ΔE.

(a) Let the entire system be in its ground state. What is its entropy? If the anergy $\Delta E = \epsilon$ is added to the system, what is its entropy? Given the definition of temperature above, what can you say about the temperature of the system if it is in the ground state?

(b) Let the total energy of the system be $2N\epsilon - \epsilon$. What is the entropy of the system? What is the entropy of the system if energy $\Delta E = \epsilon$ is added to it? If the system has energy $2N\epsilon - \epsilon$, what can you say about the temperature of the system?

Problem 2: Consider an isolated system of $N >> 1$ weakly interacting, distinct particles in equilibrium. Each particle can be in one of M states with energies $\epsilon_0, 2\epsilon_0, ..., M\epsilon_0$. Can this system exhibit negative temperatures? If so, give a value of energy corresponding to which the temperature of the system is (a) positive (b) negative. (c) If $M \longrightarrow \infty$, will the system exhibit negative temperatures? Give a physical argument.

Problem 3: Consider two systems A and B, system A consisting of $N_A >> 1$ weakly interacting particles, each of which can be in one of an infinite number of possible quantum states with energies $0, \epsilon, 2\epsilon, 3\epsilon, ...$ System B on the other hand consisting of $N_B >> 1$ weakly interacting particles, each of which can be in one of two quantum states with energies $0, \epsilon$. Initially, these systems are insulated from each other, with system A having total energy $N_A\epsilon$ and B having energy $3N_B\epsilon/4$.

(a) What can you say about the sign of the temperatures of these two systems?

(b) The systems are now made to interact with each other, till they reach equilibrium. What is the sign of the temperature of each system after equilibrium is attained?

Problem 4: The entropy function of a system is given by $S(E) = aE(E_0 - E)$, where a and E_0 are positive constants. Show that the system temperature is positive only when $E < E_0/2$. What is the system temperature for $E > E_0/2$?

1.6 Equipartition Principle

Problem 1: Consider a classical system of $N >> 1$ independent oscillators, each of which has energy given by

$$\epsilon = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

where x and p are the position and momentum of the particle. If the system is in equilibrium at temperature T, what is the molar specific heat of the system? If the expression for energy has a small correction which is not quadratic in x, what will be the qualitative change in the behaviour of the specific heat of the system?

Problem 2: Consider a weakly interacting system of particles, such that the expression for energy of any one particle consists of n terms, quadratic in position and momentum components. If classical physics is an adequate description of this system, what is the molar specific heat of the system? If the temperature of the system is progressively lowered, will the experimentally measured molar specific heat be in agreement with this result? Explain.

2 Theory of Radiation

Problem 1: Estimate the surface temperature of the red giant star Aldebaran, given that it emits radiation with maximum intensity at a wavelength of $7250\mathring{A}$. You can use the fact that the maximum intensity of solar radiation is at wavelength $5000\mathring{A}$ and corresponds to a surface temperature of about $5780°K$.

Problem 2: Electromagnetic radiation inside a cavity of volume V is in equilibrium at temperature T. If the temperature of the cavity is halved, by what factor does the pressure change? How does this compare with a classical monoatomic gas under similar conditions?

Problem 3: At what rate does radiation escape from a hole $10cm^2$ in area, in the wall of a furnace whose interior is at a temperature of $1000°K$?

Problem 4: A crystalline dielectric solid has refractive index n, assumed to be reasonably constant. The solid is at a temperature of $300°K$. Calculate the contribution of black-body radiation to its molar specific heat. Compare this with the classical molar specific heat of $3R$

Problem 5: The surface temperature of the Sun is about $5500°K$ and its radius about 7×10^8 m. The radius of the Earth is about 6.4×10^4 m and the mean distance of the Earth from the sun is about 1.5×10^{11} m. Assume that the Sun acts as a perfect black body and that the Earth absorbs all the radiation incident on it (and then re-emits it like a blackbody, ignoring greenhouse effects). Given that the Earth is in radiative equilibrium, estimate the temperature of the Earth.

Problem 6: The cosmic microwave background radiation left over from the Big Bang today fills the universe with blackbody radiation at temperature $T = 2.76°K$. What is the mean number density of photons? Use can use the following result:

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2.404$$

Problem 7: Consider blackbody radiation in equilibrium at temperature T. Let $u(\nu)d\nu$ be the energy density of the radiation in the frequency interval ν to $\nu + d\nu$ and let $\tilde{u}(\lambda)d\lambda$ be the energy density in the wavelength interval λ to $\lambda + d\lambda$. Calculate ν_{max} and λ_{max}, where ν_{max} is the frequency corresponding to which $u(\nu)$ is maximum and λ_{max} the wavelength corresponding to which $\tilde{u}(\lambda)$ is maximum. Are these related as $\lambda_{max} = c\ \nu_{max}$? Why?

Problem 8: A gas cloud in our galaxy emits radiation at a rate of $10^{27}W$. The radiation has maximum intensity at wavelength $\lambda = 10\mu m$. Assuming the cloud to be spherical and that it emits like a blackbody, estimate the diameter of the cloud.

Problem 9: Consider a hypothetical system of massless Bosonic particles which can be emitted and absorbed by matter, just like photons. The system is confined to a two-dimensional area A and is in equilibrium at temperature T. Performing an analysis similar to that for a three-dimensional photon gas, determine the temperature dependence of the energy density (energy per unit area) of the system.

Problem 10: A solid copper sphere cools at the rate of 2.8^0C per minute when its temperature is 127^0C. At what rate will a copper sphere of twice the radius cool when its temperature is 227^0C, if in both cases, the surroundings are maintained at 27^0C? Assume blackbody approximation for the calculation.

Problem 11: A 40W tungsten lamp at steady state has filament temperature 2170^0C. The effective surface area of the filament is $0.66cm^2$. What is the emissivity of the lamp, neglecting non-radiation losses? (Emissivity of a surface is defined as the ratio of the radiation energy emitted by it to that emitted by a blackbody at the same temperature).

3 Fermi Dirac and Bose Einstein Statistics

Problem 1: Seven Bosons are arranged in two compartments. The first compartment has 8 cells and the second compartment has 9 cells of equal size. What is the total number of microstates for the macrostate (3,4)?

Problem 2: Six Fermions are arranged in two compartments. The first compartment has 7 cells and the second compartment has 8 cells of equal size. What is the total number of microstates for the macrostate (2,4)?

Problem 3: Consider a system of two weakly interacting particles. Each particle can be in one of two states with energies 0 and ϵ respectively. Calculate the partition function of the system if the system is (i) Bosonic (ii) Fermionic. Calculate the mean energy of the system as a function of temperature and its value as the temperature approaches absolute zero. Give a physical interpretation of the zero temperature result.

Problem 4: Calculate the Fermi energy for Silver, given that its density is 10.5 g/cc. The atomic mass of Silver is 108 g. Assume there is one free electron per atom.

Problem 5: Given a Fermi gas, what is the mean occupation number for a state with energy $2k_BT$ above the Fermi energy?

Problem 6: The occupation number for a system of Bosons diverges for $\epsilon = \mu$ at finite T. What phenomenon does this indicate? Does this happen for a system of photons?

Problem 7: An ideal non-relativistic Fermi gas at absolute zero has Fermi energy ϵ_F, with each particle having mass m. Calculate the mean value of v_x and v_x^2, where v_x is the x component of velocity of a particle.

Problem 8: Find an expression for the Fermi energy and the average energy per electron at $0°K$ for a free electron gas of N electrons confined to a one-dimensional region of length L.

Problem 9: Consider a system of N Bosons occupying volume V. At high enough temperature T, the system behaves like a classical idea gas, such that the pressure of the system is proportional to T. If the temperature is such that the system is strongly degenerate, given that a certain fraction of atoms is in the ground state (and does not contribute to pressure), what power of temperature do you expect the pressure to be proportionate to? Explain.

Problem 10: Consider a free electron gas consisting of N electrons occupying volume V. At high enough temperature T, the system behaves like a classical idea gas, such that the pressure of the system is proportional to T. At absolute zero, the system exerts a non-zero pressure, the Fermi pressure. If the temperature of the system is such that the system is strongly degenerate, given that a certain fraction of electrons are excited above the Fermi energy (and assuming that this fraction exerts pressure just like a classical gas), what power of temperature do you expect the increase in pressure relative to absolute zero to be proportionate to? Explain.

Problem 11: Consider a system of non-interacting quantum particles in three dimensions with dispersion relation $\epsilon \propto k^s$ where ϵ is energy and $\vec{k}$ is the wave-vector, where s is an integer. To what power of ϵ is the density of states proportional to?

Problem 12: Consider a system of N weakly interacting non-relativistic Bosons confined to a two-dimensional region of area A. Repeating the standard analysis in three dimensions, test whether Bose-Einstein condensation occurs in this system at a non-zero temperature.

Problem 13: A white dwarf star has mass $M = 2 \times 10^{30}$kg and radius $R = 7 \times 10^6$m. What is the degeneracy pressure of the electron gas in the star?

Problem 14: Show that for a stable white dwarf star involving a strongly egenerate and non-relativistic electron gas, the Mass-Radius relationship is $M^{1/3}R =$ constant. Assume uniform mass density.

Problem 15: An ideal gas of Rb^{87} particles at 100^0K is compressed isothermally. What is the number density at which the Bose-Einstein condensation starts?

Problem 16: An ideal Bose gas with spinless particles of mass 6.65×10^{-27}kg is at a particle density $n = 10^{26} m^{-3}$. What is the percentage of particles in the ground state if the gas temperature is $T = 0.043^0$K? What is the entropy of the gas at this temperature?